Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

II.—*An Essay towards a dynamical Theory of crystalline Reflexion and Refraction. By* JAMES MAC CULLAGH, *Fellow of Trinity College, Dublin.*

Read December 9th, 1839.

SECT. I.—INTRODUCTORY OBSERVATIONS.—EQUATION OF MOTION.

NEARLY three years ago I communicated to this Academy* the laws by which the vibrations of light appear to be governed in their reflexion and refraction at the surfaces of crystals. These laws—remarkable for their simplicity and elegance, as well as for their agreement with exact experiments—I obtained from a system of hypotheses which were opposed, in some respects, to notions previously received, and were not bound together by any known principles of mechanics, the only evidence of their truth being the truth of the results to which they led. On that occasion, however, I observed that the hypotheses were not independent of each other; and soon afterwards I proved that the laws of reflexion at the surface of a crystal are connected, in a very singular way, with the laws of double refraction, or of propagation in its interior; from which I was led to infer that "all these laws and hypotheses have a common source in other and more intimate laws which remain to be discovered;" and that "the next step in physical optics would probably lead to those higher and more elementary principles by which the laws of reflexion and the laws of propagation are linked together as parts of the same system."† This step has since been made, and these anticipations have been realised. In the present paper I propose to supply the link between the two sets of laws by means of a very simple theory, depending on certain special assumptions, and employing the usual methods of analytical dynamics.

---

* In a paper "on the Laws of crystalline Reflexion and Refraction."—Transactions of the Royal Irish Academy, vol. xviii. p. 31.

† Ibid. p. 53, note. The note here referred to was added some time after the paper itself was read.

In this theory, the two kinds of laws, being traced from a common origin, are at once connected with each other and severally explained; and it may be observed, that the explanation of each, as well as the source of their connexion, is now made known for the first time. For though the laws of crystalline propagation have attracted much attention during the period which has elapsed since they were discovered by Fresnel,* they have hitherto resisted every attempt that has been made to account for them by dynamical reasonings; and the laws of reflexion, when recently discovered, were apparently still more difficult to be reached by such considerations. Nothing can be easier, however, than the process by which both systems of laws are now deduced from the same simple principles.

The assumptions on which the theory rests are these:—*First,* that the density of the luminiferous ether is a constant quantity; in which it is implied that this density is unchanged either by the motions which produce light or by the presence of material particles, so that it is the same within all bodies as in free space, and remains the same during the most intense vibrations. *Second,* that the vibrations in a plane wave are rectilinear, and that, while the plane of the wave moves parallel to itself, the vibrations continue parallel to a fixed right line, the direction of this right line and the direction of a normal to the wave being functions of each other. This supposition holds in all known crystals, except quartz, in which the vibrations are elliptical.

Concerning the peculiar constitution of the ether we know nothing, and shall suppose nothing, except what is involved in the foregoing assumptions. But with respect to its physical condition generally, we shall admit, as is most natural, that a vast number of ethereal particles are contained in the differential element of volume; and, for the present, we shall consider the mutual action of these particles to be sensible only at distances which are insensible when compared with the length of a wave.

By putting together the assumptions we have made, it will appear that when a system of plane waves disturbs the ether, the vibrations are transversal, or parallel to the plane of waves. For all the particles situated in a plane parallel to the waves are displaced, from their positions of rest, through equal spaces in parallel directions; and therefore if we conceive a closed surface of any form, including any volume great or small, to be described in the quiescent ether, and then all its points to partake of the motion imparted by the waves, any slice cut out of that volume, by a pair of planes parallel to the wave-plane and indefinitely near each other, can have nothing but its thickness altered by the displacements; and since the assumed preservation of density requires that the volume of the slice should not be altered, nor consequently its thickness, it follows that the displacements must be in the plane of the slice, that is to say, they must be parallel to the wave-plane. And conversely, when this condition is fulfilled, it is obvious that the entire volume, bounded by the arbitrary surface above described, will remain constant during the motion, while the surface itself will always contain within it the very same ethereal particles which it enclosed in the state of rest; and all this will be accurately true, no matter how great may be the magnitude of the displacements.

---

* These laws were published in his Memoir on Double Refraction.—Mémoires de l'Institut, tom. vii. p. 45.

Let x, y, z be the rectangular coordinates of a particle before it is disturbed, and $x+\xi$, $y+\eta$, $z+\zeta$ its coordinates at the time t, the displacements ξ, η, ζ being functions of x, y, z and t. Let the ethereal density, which is the same in all media, be regarded as unity, so that $dxdydz$ may, at any instant, represent indifferently either the element of volume or of mass. Then the equation of motion will be of the form

$$\iiint dxdydz\left(\frac{d^2\xi}{dt^2}\,\delta\xi+\frac{d^2\eta}{dt^2}\,\delta\eta+\frac{d^2\zeta}{dt^2}\,\delta\zeta\right)=\iiint dxdydz\,\delta\mathrm{V}, \qquad (1)$$

where V is some function depending on the mutual actions of the particles. The integrals are to be extended over the whole volume of the vibrating medium, or over all the media, if there be more than one.

Setting out from this equation, which is the general formula of dynamics applied to the case that we are considering, we perceive that our chief difficulty will consist in the right determination of the function V; for if that function were known, little more would be necessary, in order to arrive at all the laws which we are in search of, than to follow the rules of analytical mechanics, as they have been given by Lagrange. The determination of V will, of course, depend on the assumptions above stated respecting the nature of the ethereal vibrations; but, before we proceed further, it seems advisable to introduce certain lemmas, for the purpose of abridging this and the subsequent investigations.

SECT. II.—LEMMAS.

Lemma I. Let a right line making with three rectangular axes the angles α, β, γ, be perpendicular to two other right lines which make with the same axes the angles α', β', γ' and α'', β'', γ'' respectively, and which are inclined to each other at an angle denoted by θ; then it is easy to prove that

$$\begin{aligned}\sin\theta\cos\alpha &= \cos\beta'\cos\gamma'' - \cos\beta''\cos\gamma',\\ \sin\theta\cos\beta &= \cos\gamma'\cos\alpha'' - \cos\gamma''\cos\alpha',\\ \sin\theta\cos\gamma &= \cos\alpha'\cos\beta'' - \cos\alpha''\cos\beta';\end{aligned} \qquad (\text{A})$$

supposing the first right line to be prolonged in the proper direction from the origin, in order that the opposite members of any one of these equations may have the same sign, as well as the same magnitude.

If the last two right lines be perpendicular to each other, we have $\sin\theta = 1$, and the formulæ become

$$\begin{aligned}\cos\alpha &= \cos\beta'\cos\gamma'' - \cos\beta''\cos\gamma',\\ \cos\beta &= \cos\gamma'\cos\alpha'' - \cos\gamma''\cos\alpha',\\ \cos\gamma &= \cos\alpha'\cos\beta'' - \cos\alpha''\cos\beta';\end{aligned} \qquad (\text{B})$$

but in this case the three right lines are perpendicular to each other, and therefore we have, in like manner,

$$\begin{aligned}\cos\alpha' &= \cos\beta''\cos\gamma - \cos\beta\cos\gamma'',\\ \cos\beta' &= \cos\gamma''\cos\alpha - \cos\gamma\cos\alpha'',\\ \cos\gamma' &= \cos\alpha''\cos\beta - \cos\alpha\cos\beta'';\end{aligned} \qquad (\text{B}')$$

and also,

$$\begin{aligned}\cos\alpha'' &= \cos\beta\cos\gamma' - \cos\beta'\cos\gamma,\\ \cos\beta'' &= \cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha,\\ \cos\gamma'' &= \cos\alpha\cos\beta' - \cos\alpha'\cos\beta.\end{aligned} \qquad (\text{B}'')$$

The last three groups of formulæ will still be true, if we suppose the first right line to make with the axes the angles α, α', α'', the second the angles β, β', β'', and the third the angles γ, γ', γ''.

Lemma II. Let ξ, η, ζ denote, as before, the displacements of a particle whose initial coordinates are x, y, z; and after putting

$$\mathrm{X} = \frac{d\eta}{dz} - \frac{d\zeta}{dy}, \quad \mathrm{Y} = \frac{d\zeta}{dx} - \frac{d\xi}{dz}, \quad \mathrm{Z} = \frac{d\xi}{dy} - \frac{d\eta}{dx}, \tag{c}$$

suppose the axes of coordinates, still remaining rectangular, to have their directions changed in space, whereby the quantities X, Y, Z will be changed into X', Y', Z', answering to the new coordinates x', y', z', and to the new displacements ξ', η', ζ'; then will the quantities X', Y', Z' be connected with X, Y, Z by the very same relations which connect the coordinates x', y', z' with x, y, z, or the displacements ξ', η', ζ' with ξ, η, ζ.

That is to say, if the axes of x, y, z make with the axis of x' the angles α, β, γ, with the axis of y' the angles α', β', γ', and with the axis of z' the angles α'', β'', γ'' respectively, we shall have

$$\begin{aligned} \mathrm{X} &= \mathrm{X}' \cos\alpha + \mathrm{Y}' \cos\alpha' + \mathrm{Z}' \cos\alpha'', \\ \mathrm{Y} &= \mathrm{X}' \cos\beta + \mathrm{Y}' \cos\beta' + \mathrm{Z}' \cos\beta'', \\ \mathrm{Z} &= \mathrm{X}' \cos\gamma + \mathrm{Y}' \cos\gamma' + \mathrm{Z}' \cos\gamma'', \end{aligned} \tag{D}$$

and

$$\begin{aligned} \mathrm{X}' &= \mathrm{X} \cos\alpha + \mathrm{Y} \cos\beta + \mathrm{Z} \cos\gamma, \\ \mathrm{Y}' &= \mathrm{X} \cos\alpha' + \mathrm{Y} \cos\beta' + \mathrm{Z} \cos\gamma', \\ \mathrm{Z}' &= \mathrm{X} \cos\alpha'' + \mathrm{Y} \cos\beta'' + \mathrm{Z} \cos\gamma''; \end{aligned} \tag{D$'$}$$

just as we have, for example,

$$\begin{aligned} \xi &= \xi' \cos\alpha + \eta' \cos\alpha' + \zeta' \cos\alpha'', \\ \eta &= \xi' \cos\beta + \eta' \cos\beta' + \zeta' \cos\beta'', \\ \zeta &= \xi' \cos\gamma + \eta' \cos\gamma' + \zeta' \cos\gamma'', \end{aligned} \tag{d}$$

and

$$\begin{aligned} x' &= x \cos\alpha + y \cos\beta + z \cos\gamma, \\ y' &= x \cos\alpha' + y \cos\beta' + z \cos\gamma', \\ z' &= x \cos\alpha'' + y \cos\beta'' + z \cos\gamma''. \end{aligned} \tag{d$'$}$$

For, the change of the independent variables x, y, z into x', y', z' gives us the equations

$$\frac{d\eta}{dz}=\frac{d\eta}{dx'}\frac{dx'}{dz}+\frac{d\eta}{dy'}\frac{dy'}{dz}+\frac{d\eta}{dz'}\frac{dz'}{dz},$$

$$\frac{d\zeta}{dy}=\frac{d\zeta}{dx'}\frac{dx'}{dy}+\frac{d\zeta}{dy'}\frac{dy'}{dy}+\frac{d\zeta}{dz'}\frac{dz'}{dy},$$

in the right-hand members of which we have to substitute the values of the differential coefficients obtained from (d) and (d′). Thus we get

$$\begin{aligned}\frac{d\eta}{dz}=&\left(\frac{d\xi'}{dx'}\cos\beta+\frac{d\eta'}{dx'}\cos\beta'+\frac{d\zeta'}{dx'}\cos\beta''\right)\cos\gamma\\&+\left(\frac{d\xi'}{dy'}\cos\beta+\frac{d\eta'}{dy'}\cos\beta'+\frac{d\zeta'}{dy'}\cos\beta''\right)\cos\gamma'\\&+\left(\frac{d\xi'}{dz'}\cos\beta+\frac{d\eta'}{dz'}\cos\beta'+\frac{d\zeta'}{dz'}\cos\beta''\right)\cos\gamma'',\\\frac{d\zeta}{dy}=&\left(\frac{d\xi'}{dx'}\cos\gamma+\frac{d\eta'}{dx'}\cos\gamma'+\frac{d\zeta'}{dx'}\cos\gamma''\right)\cos\beta\\&+\left(\frac{d\xi'}{dy'}\cos\gamma+\frac{d\eta'}{dy'}\cos\gamma'+\frac{d\zeta'}{dy'}\cos\gamma''\right)\cos\beta'\\&+\left(\frac{d\xi'}{dz'}\cos\gamma+\frac{d\eta'}{dz'}\cos\gamma'+\frac{d\zeta'}{dz'}\cos\gamma''\right)\cos\beta'';\end{aligned}$$

and when we subtract these equations, attending to the formulæ in Lemma I., we find

$$\frac{d\eta}{dz}-\frac{d\zeta}{dy}=\left(\frac{d\eta'}{dz'}-\frac{d\zeta'}{dy'}\right)\cos\alpha+\left(\frac{d\zeta'}{dx'}-\frac{d\xi'}{dz'}\right)\cos\alpha'+\left(\frac{d\xi'}{dy'}-\frac{d\eta'}{dx'}\right)\cos\alpha'',$$

or simply,

$$\mathrm{X}=\mathrm{X}'\cos\alpha+\mathrm{Y}'\cos\alpha'+\mathrm{Z}'\cos\alpha'',$$

which is the first of formulæ (D). And in like manner the others may be proved.

The same things will obviously hold with respect to quantities derived from X, Y, Z in the same way that these are derived from ξ, η, ζ. That is, if we put

$$\mathrm{X}_{,}=\frac{d\mathrm{Y}}{dz}-\frac{d\mathrm{Z}}{dy},\quad \mathrm{Y}_{,}=\frac{d\mathrm{Z}}{dx}-\frac{d\mathrm{X}}{dz},\quad \mathrm{Z}_{,}=\frac{d\mathrm{X}}{dy}-\frac{d\mathrm{Y}}{dx},$$

and then suppose the axes of coordinates to be changed, the formulæ for the transformation of the quantities $\mathrm{X}_{,}$, $\mathrm{Y}_{,}$, $\mathrm{Z}_{,}$, will be similar to those for the transformation of the coordinates themselves. The like will be true of the quantities $\text{X}_{,,}$, $\text{Y}_{,,}$, $\text{Z}_{,,}$, if we put

$$\text{X}_{,,}=\frac{d\text{Y}_{,}}{dz}-\frac{d\text{Z}_{,}}{dy},\quad \text{Y}_{,,}=\frac{d\text{Z}_{,}}{dx}-\frac{d\text{X}_{,}}{dz},\quad \text{Z}_{,,}=\frac{d\text{X}_{,}}{dy}-\frac{d\text{Y}_{,}}{dx};$$

and so on successively.

It is to be observed that, in this Lemma, the displacement is not limited by any restriction whatever. Each of its components may be any function of the coordinates. But the displacements produced by a system of plane waves are restricted by our definition of such waves; they must be the same for all particles situated in the same wave-plane. If the waves be parallel, for instance, to the plane of $x'\,y'$, the quantities ξ', η', ζ' will be independent of the coordinates x', y', and will be functions of z' only. This consideration reduces formulæ (D) to the following,

$$\begin{aligned}\text{X}&=\frac{d\eta'}{dz'}\cos\alpha-\frac{d\xi'}{dz'}\cos\alpha',\\ \text{Y}&=\frac{d\eta'}{dz'}\cos\beta-\frac{d\xi'}{dz'}\cos\beta',\\ \text{Z}&=\frac{d\eta'}{dz'}\cos\gamma-\frac{d\xi'}{dz'}\cos\gamma',\end{aligned}\qquad(\text{E})$$

in which it is remarkable that the normal displacement ζ' does not appear. If $\xi'=0$, these formulæ become

$$\text{X}=\frac{d\eta'}{dz'}\cos\alpha,\quad \text{Y}=\frac{d\eta'}{dz'}\cos\beta,\quad \text{Z}=\frac{d\eta'}{dz'}\cos\gamma;\qquad(\text{F})$$

or if $\eta'=0$, then we have

$$\text{X}=-\frac{d\xi'}{dz'}\cos\alpha',\quad \text{Y}=-\frac{d\xi'}{dz'}\cos\beta',\quad \text{Z}=-\frac{d\xi'}{dz'}\cos\gamma'.\qquad(\text{F}')$$

Lemma III. If, in an ellipsoid whose semiaxes are equal to a, b, c, there be two rectangular diameters, one making with the semiaxes the angles α, β, γ, and the other the angles α', β', γ', such as to satisfy the condition

$$\frac{\cos\alpha\cos\alpha'}{a^2}+\frac{\cos\beta\cos\beta'}{b^2}+\frac{\cos\gamma\cos\gamma'}{c^2}=0,\qquad(\text{G})$$

these diameters will be the axes of the ellipse in which their plane intersects the ellipsoid.

For, the above condition expresses that either diameter is parallel to the tangent plane at the extremity of the other; they are therefore conjugate diameters of the elliptic section, and hence, as they are at right angles to each other, they must be its axes.

If the semiaxes of the ellipsoid be represented by $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, the equation of condition will become

$$a^2 \cos \alpha \cos \alpha' + b^2 \cos \beta \cos \beta' + c^2 \cos \gamma \cos \gamma' = 0. \qquad (\text{G}')$$

Lemma IV. Let s, s' be the lengths of perpendiculars let fall from the centre of an ellipsoid upon any two tangent planes, and r, r' the lengths of radii drawn to the respective points of contact. Then putting ω for the angle between the directions of r and s', and ω' for the angle between the directions of r' and s, we shall have

$$rs \cos \omega = r' s' \cos \omega'.$$

For if the semiaxes of the ellipsoid, having their lengths denoted by a, b, c, make with the direction of s the angles α, β, γ, and with that of s' the angles α', β', γ'; with the direction of r the angles α_0, β_0, γ_0, and with that of r' the angles α_1, β_1, γ_1; there will exist the relations

$$\begin{array}{lll} a^2 \cos \alpha = rs \cos \alpha_0, & b^2 \cos \beta = rs \cos \beta_0, & c^2 \cos \gamma = rs \cos \gamma_0, \\ a^2 \cos \alpha' = r' s' \cos \alpha_1, & b^2 \cos \beta' = r' s' \cos \beta_1, & c^2 \cos \gamma' = r' s' \cos \gamma_1, \end{array}$$

by one set of which the quantity

$$a^2 \cos \alpha \cos \alpha' + b^2 \cos \beta \cos \beta' + c^2 \cos \gamma \cos \gamma'$$

will be converted into

$$rs (\cos \alpha_0 \cos \alpha' + \cos \beta_0 \cos \beta' + \cos \gamma_0 \cos \gamma') = rs \cos \omega,$$

and by the other set into

$$r' s' (\cos \alpha_1 \cos \alpha + \cos \beta_1 \cos \beta + \cos \gamma_1 \cos \gamma) = r' s' \cos \omega';$$

so that we shall get

$$rs \cos \omega = r' s' \cos \omega' = a^2 \cos \alpha \cos \alpha' + b^2 \cos \beta \cos \beta' + c^2 \cos \gamma \cos \gamma'. \qquad (\text{H})$$

Corollary. When the condition

$$a^2 \cos \alpha \cos \alpha' + b^2 \cos \beta \cos \beta' + c^2 \cos \gamma \cos \gamma' = 0 \qquad (1)$$

is satisfied, each of the angles ω, ω' is a right angle. Let us suppose, at the same time, that the direction of s is perpendicular to that of s'. Then will the directions of s and r' coincide with the axes of the ellipse in which their plane intersects the ellipsoid; for s is perpendicular to r' and parallel to the tangent plane at its extremity. The directions of s' and r, in the same manner, will coincide with the axes of another elliptic section.

SECT. III.—DETERMINATION OF THE FUNCTION ON WHICH THE MOTION DEPENDS. PRINCIPAL AXES OF A CRYSTAL.

We come now to investigate the particular form which must be assigned to the function V, in order that the formula (1) may represent the motions of the ethereal medium. For this purpose conceive the plane of $x'y'$ to be parallel to a system of plane waves whose vibrations are entirely transversal and parallel to the axis of y', so that $\xi' = 0$, $\zeta' = 0$. Imagine an elementary parallelepiped $dx'dy'dz'$, having its edges parallel to the axes of x', y', z', to be described in the ether when at rest, and then all its points to move according to the same law as the ethereal particles which compose it. The faces of the parallelepiped which are perpendicular to the edge dz' will be shifted, each in its own plane, in a direction parallel to the axis of y', but their displacements will be unequal, and will differ by $d\eta'$, so that the edges connecting their corresponding angles will no longer be parallel to the axis of z', but will be inclined to it at an angle κ whose tangent is $\frac{d\eta'}{dz'}$.

Now the function V can only depend upon the directions of the axes of x', y', z' with respect to fixed lines in the crystal, and upon the angle κ, which measures the change of form produced in the parallelepiped by vibration. This is the most general supposition which can be made concerning it. Since, however, by our second assumption, any one of these directions, suppose that of x', determines the other two, we may regard V as depending on the angle κ and on the direction of the axis of x' alone. But from the equations (F) it is manifest that the angle κ and the angles which the axis of x' makes with the fixed axes of x, y, z, are all known when the quantities X, Y, Z are known. Consequently V is a function of X, Y, Z.

Supposing the angle κ to be very small, the quantities X, Y, Z will also be very small; and if V be expanded according to the powers of these quantities, we shall have

$$V = K + AX + BY + CZ + DX^2 + EY^2 + FZ^2 + GYZ + HXZ + IXY + \text{etc.},$$

the quantities K, A, B, C, D, &c., being constant. But in the state of equilibrium the value of δV ought to be nothing, in whatever way the position of the system be varied; that is to say, when the displacements ξ, η, ζ, and consequently the quantities X, Y, Z, are supposed to vanish, the quantity

$$\delta V = A\delta X + B\delta Y + C\delta Z + 2DX\delta X + \text{etc.},$$

ought also to vanish independently of the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, or, which comes to the same thing, independently of δX, δY, δZ. Hence* we must have $A = 0$, $B = 0$, $C = 0$; and therefore, if we neglect terms of the third and higher dimensions, we may conclude that the variable part of V is a homogeneous function of the second degree, containing, in its general form, the squares and products of X, Y, Z, with six constant coefficients.

Of these coefficients, the three which multiply the products of the variables may always be made to vanish by changing the directions of the axes of x, y, z. For this is a known property of functions of the second degree, when the coordinates are the variables; and we have shown, in Lemma II., that the quantities X, Y, Z, are transformed by the very same relations as the coordinates themselves. Therefore, in every crystal there exist three rectangular axes, with respect to which the function V contains only the squares of X, Y, Z; and as it will presently appear that the coefficients of the squares must all be negative, in order that the velocity of propagation may never become imaginary, we may consequently write, with reference to these axes,

$$V = -\tfrac{1}{2}(a^2 X^2 + b^2 Y^2 + c^2 Z^2), \qquad (2)$$

omitting the constant K as having no effect upon the motion.

The axes of coordinates, in this position, are the *principal axes* of the crystal, and are commonly known by the name of *axes of elasticity*. Thus the existence of these axes is proved without any hypothesis respecting the arrangement of the particles of the medium. The constants a, b, c are the three principal velocities of propagation, as we shall see in the next section.

---

* See the reasoning of Lagrange in an analogous case, Mécanique Analytique, tom. i. p. 68.

Having arrived at the value of V, we may now take it for the starting point of our theory, and dismiss the assumptions by which we were conducted to it. Supposing therefore, in the first place, that a plane wave passes through a crystal, we shall seek the laws of its motion from equations (1) and (2), which contain everything that is necessary for the solution of the problem. The laws of propagation, as they are called, will in this way be deduced, and they will be found to agree exactly, so far as *magnitudes* are concerned, with those discovered by Fresnel; but the *direction* of the vibrations in a polarised ray will be different from that assigned by him. In the second place, we shall investigate the conditions which are fulfilled when light passes out of one medium into another, and we shall thus obtain the laws of reflexion and refraction at the surface of a crystal.

SECT. IV.—PROPAGATION OF LIGHT IN A CRYSTALLIZED MEDIUM.—LAWS OF FRESNEL.—ALTERATION REQUIRED TO BE MADE IN THEM.—WAVE-SURFACE, INDEX-SURFACE, AND THEIR PROPERTIES.

The principal axes of the crystal being the axes of x, y, z, we have, by equation (2),

$$-\delta V = a^2\mathrm{X}\delta\mathrm{X} + b^2\mathrm{Y}\delta\mathrm{Y} + c^2\mathrm{Z}\delta\mathrm{Z}, \tag{3}$$

or, by taking the variations from formulæ (c), and interchanging the characteristics d and δ,

$$-\delta V = a^2\mathrm{X}\left(\frac{d\delta\eta}{dz} - \frac{d\delta\zeta}{dy}\right) + b^2\mathrm{Y}\left(\frac{d\delta\zeta}{dx} - \frac{d\delta\xi}{dz}\right) + c^2\mathrm{Z}\left(\frac{d\delta\xi}{dy} - \frac{d\delta\eta}{dx}\right);$$

and if we substitute this value in equation (1), and then integrate by parts the right-hand member, in order to get rid of the differential coefficients of the variations, we shall obtain

$$\iiint dxdydz\left(\frac{d^2\xi}{dt^2}\,\delta\xi + \frac{d^2\eta}{dt^2}\,\delta\eta + \frac{d^2\zeta}{dt^2}\,\delta\zeta\right) =$$

$$\iint dydz\,(c^2\mathrm{Z}\delta\eta - b^2\mathrm{Y}\delta\zeta) + \iint dxdz\,(a^2\mathrm{X}\delta\zeta - c^2\mathrm{Z}\delta\xi) + \iint dxdy\,(b^2\mathrm{Y}\delta\xi - a^2\mathrm{X}\delta\eta) \tag{4}$$
$$+ \iiint dxdydz\left\{\left(c^2\frac{d\mathrm{Z}}{dy} - b^2\frac{d\mathrm{Y}}{dz}\right)\delta\xi + \left(a^2\frac{d\mathrm{X}}{dz} - c^2\frac{d\mathrm{Z}}{dx}\right)\delta\eta + \left(b^2\frac{d\mathrm{Y}}{dx} - a^2\frac{d\mathrm{X}}{dy}\right)\delta\zeta\right\}.$$

But as the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$ are arbitrary and independent, this equation cannot hold unless the double integrals, which relate to the limits of the system, reduce themselves to zero, leaving the equality to subsist, independently of the variations, between the triple integrals alone. Equating, therefore, the coefficients of the corresponding variations in the triple integrals, we get

$$\begin{aligned}
\frac{d^2\xi}{dt^2} &= c^2\frac{d\mathrm{Z}}{dy} - b^2\frac{d\mathrm{Y}}{dz},\\
\frac{d^2\eta}{dt^2} &= a^2\frac{d\mathrm{X}}{dz} - c^2\frac{d\mathrm{Z}}{dx},\\
\frac{d^2\zeta}{dt^2} &= b^2\frac{d\mathrm{Y}}{dx} - a^2\frac{d\mathrm{X}}{dy},
\end{aligned} \qquad (5)$$

which are the equations of propagation, giving the expression for the accelerating force parallel to each axis of coordinates.

When there is a single medium extending indefinitely on all sides, the conditions relative to the limits are of no importance, and we have only to consider the equations (5), from which we shall now deduce the laws by which a system of plane waves is propagated.

Supposing the waves to be parallel to the plane of $x'\,y'$, the displacements will be functions of z' only, and if ψ be any function of the displacements, we shall have, by formulæ (d'),

$$\frac{d\psi}{dx} = \frac{d\psi}{dz'}\frac{dz'}{dx} = \frac{d\psi}{dz'}\cos\alpha'', \quad \frac{d\psi}{dy} = \frac{d\psi}{dz'}\cos\beta'', \quad \frac{d\psi}{dz} = \frac{d\psi}{dz'}\cos\gamma'',$$

so that the equations (5) may be written

$$\begin{aligned}
\frac{d^2\xi}{dt^2} &= c^2\frac{d\mathrm{Z}}{dz'}\cos\beta'' - b^2\frac{d\mathrm{Y}}{dz'}\cos\gamma'',\\
\frac{d^2\eta}{dt^2} &= a^2\frac{d\mathrm{X}}{dz'}\cos\gamma'' - c^2\frac{d\mathrm{Z}}{dz'}\cos\alpha'',\\
\frac{d^2\zeta}{dt^2} &= b^2\frac{d\mathrm{Y}}{dz'}\cos\alpha'' - a^2\frac{d\mathrm{X}}{dz'}\cos\beta'';
\end{aligned}$$

and when we combine these with the following,

$$\frac{d^2\xi'}{dt^2} = \frac{d^2\xi}{dt^2}\cos\alpha + \frac{d^2\eta}{dt^2}\cos\beta + \frac{d^2\zeta}{dt^2}\cos\gamma,$$

$$\frac{d^2\eta'}{dt^2} = \frac{d^2\xi}{dt^2}\cos\alpha' + \frac{d^2\eta}{dt^2}\cos\beta' + \frac{d^2\zeta}{dt^2}\cos\gamma',$$

$$\frac{d^2\zeta'}{dt^2} = \frac{d^2\xi}{dt^2}\cos\alpha'' + \frac{d^2\eta}{dt^2}\cos\beta'' + \frac{d^2\zeta}{dt^2}\cos\gamma'',$$

attending to the relations (B), (B′), we find

$$\frac{d^2\xi'}{dt^2} = -\left(a^2\frac{d\mathrm{X}}{dz'}\cos\alpha' + b^2\frac{d\mathrm{Y}}{dz'}\cos\beta' + c^2\frac{d\mathrm{Z}}{dz'}\cos\gamma'\right),$$

$$\frac{d^2\eta'}{dt^2} = a^2\frac{d\mathrm{X}}{dz'}\cos\alpha + b^2\frac{d\mathrm{Y}}{dz'}\cos\beta + c^2\frac{d\mathrm{Z}}{dz'}\cos\gamma,$$

$$\frac{d^2\zeta'}{dt^2} = 0;$$

from which it appears that there is no accelerating force in the direction of a normal* to the wave, and consequently no vibration in that direction. Introducing now the values of X, Y, Z from formulæ (E), the first two of these equations become

$$\left.\begin{aligned}\frac{d^2\xi'}{dt^2} &= (a^2\cos^2\alpha' + b^2\cos^2\beta' + c^2\cos^2\gamma')\frac{d^2\xi'}{dz'^2}\\ &\quad-(a^2\cos\alpha\cos\alpha' + b^2\cos\beta\cos\beta' + c^2\cos\gamma\cos\gamma')\frac{d^2\eta'}{dz'^2},\\ \frac{d^2\eta'}{dt^2} &= (a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma)\frac{d^2\eta'}{dz'^2}\\ &\quad-(a^2\cos\alpha\cos\alpha' + b^2\cos\beta\cos\beta' + c^2\cos\gamma\cos\gamma')\frac{d^2\xi'}{dz'^2}.\end{aligned}\right\} \quad (6)$$

But as the axes of x', y' are arbitrarily taken in the plane of $x'\,y'$, we may subject their directions to the condition

$$a^2\cos\alpha\cos\alpha' + b^2\cos\beta\cos\beta' + c^2\cos\gamma\cos\gamma' = 0; \quad (7)$$

---

* In the ingenious, but altogether unsatisfactory theory, by which Fresnel has endeavoured to account for his beautiful laws, the direction of the elastic force brought into play by the displacements of the ethereal molecules is, in general, inclined to the plane of the wave. He supposes, however, that the force normal to that plane does not produce any appreciable effect, by reason of the great resistance which the ether offers to compression.—Mémoires de l'Institut, tom. vii. p. 78.

and then, if we put

$$s^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma, \qquad (8)$$
$$s'^2 = a^2 \cos^2\alpha' + b^2 \cos^2\beta' + c^2 \cos^2\gamma',$$

the equations (6) will be reduced to the well-known form

$$\frac{d^2\xi'}{dt^2} = s'^2 \frac{d^2\xi'}{dz'^2}, \qquad \frac{d^2\eta'}{dt^2} = s^2 \frac{d^2\eta'}{dz'^2}. \qquad (9)$$

This result shows that, when the directions of x' and y' fulfil the condition (7), the vibrations ξ' and η' are propagated independently of each other, the former with the velocity s', the latter with the velocity s. The vibrations must therefore be parallel exclusively to one or other of these directions, else the system of waves will split into two systems, one vibrating parallel to x', the other parallel to y'.

When the plane of the wave is parallel to one of the principal axes, it is easy to infer that the vibrations must be either parallel or perpendicular to that axis, and that, in the latter case, the velocity of propagation is constant, being equal to a, b, or c, according as the wave is parallel to the axis of x, y, or z. These constants are therefore called the principal velocities of propagation; and we now perceive the reason of the negative sign in equation (2), for if any of the terms in the right-hand member of that equation were positive, the corresponding velocity would be imaginary.

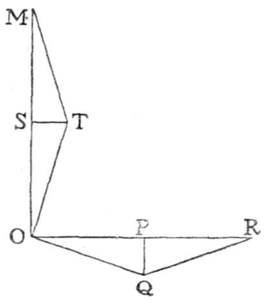
According to Fresnel, the wave which is propagated with the velocity a has its vibrations not perpendicular to the axis of x, but parallel to it; and it is to be observed that a difference of the same character distinguishes his views, throughout, from the results of the present theory. It will appear in fact, by what immediately follows, that the equations (7), (8), (9), express exactly the laws of Fresnel, provided the quantities ξ' and η', in the equations (9), be interchanged. To make these laws agree with our theory, it is therefore necessary to alter them in one particular, and in one only; it is necessary to suppose that the direction of the vibrations is always perpendicular to that assigned by Fresnel. And since, in order to make his views agree with the phenomena, Fresnel was obliged to say that, in an ordinary medium, the vibrations of a ray polarised in a certain plane are *perpendicular* to that plane, it is clear that, on the present principles, we must come to a different conclusion, and say that the vibrations of a polarised ray are *parallel* to its plane of polarisation.

Conceive an ellipsoid with its centre at O, the common origin of the coordinates x, y, z, x', y', z'; and let its semiaxes be parallel to x, y, z, their lengths being equal to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively. From the identity of the condition (7) with that marked (G′) in Lemma III., it is evident that the directions of x' and y', when they are the two directions of vibration, coincide with the axes of the ellipse in which the plane of $x'y'$ intersects the ellipsoid; and if the right line OR, meeting the ellipsoid in R, be the direction of x', we have

$$\frac{1}{(\mathrm{OR})^2} = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma,$$

or, by (8),

$$\frac{1}{\mathrm{OR}} = s,$$



so that OR is the reciprocal of the velocity with which the vibrations parallel to y' are propagated. Thus we see that the vibrations parallel to either semiaxis of the elliptic section are propagated with a velocity which is measured by the reciprocal of the other semiaxis.

Again, conceive an ellipsoid with its centre at O, and its semiaxes parallel to x, y, z, as before, but equal to a, b, c respectively. Let this ellipsoid be touched in the point Q by a plane which cuts OR perpendicularly in P, and draw the right lines OQ, PQ. Then as the condition (7) is identical with that marked (I) in the corollary to Lemma IV., it follows that Oy' (if we so call the direction of y') is perpendicular to OQ, and also that Oy' and OQ coincide with the axes of the elliptic section made in this ellipsoid by the plane QOy', just as Oy' and OR coincide with the axes of the section ROy' in the first ellipsoid. The plane QOR is therefore perpendicular to Oy' and to the plane of the wave. Moreover, we have

$$(\mathrm{OP})^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma = s^2,$$

so that OP is the reciprocal of OR, and is equal to the velocity s with which the wave is propagated when its vibrations are parallel to Oy'.

Now let the figure TOSM be equal in all respects to QOPR, but in a position perpendicular to it, so that if QOPR were turned round in its own plane through a right angle, the point O being fixed, the points Q, P, R would fall upon T, S, M respectively; and supposing the wave-plane $\mathrm{RO}y'$ to take various positions passing through O, imagine a construction similar to the preceding one to be always made by means of the two ellipsoids. Then while the points R and Q describe the ellipsoids, the points M and T describe two biaxal* surfaces reciprocal to each other, the latter surface being touched† in the point T by a plane which cuts OM perpendicularly in S. But this plane is parallel to the central wave-plane $\mathrm{RO}y'$, and distant from it by an interval OS (= OP) which represents the velocity of the wave; and as the surface whose tangent planes possess this property is, by definition, the *wave-surface* of the crystal, it is obvious that the point T describes the wave-surface. The radius OT, drawn to the point of contact, is then, by the theory of waves, the direction of the *ray* which belongs to the wave $\mathrm{RO}y'$, and the length OT represents the velocity of light along the ray. As to the surface described by the point M, it is that which I have called the *surface of indices*, or the *index-surface*,§ because its radius OM, being the reciprocal of OS, represents the index of refraction, or the ratio of the sine of the angle of incidence to the sine of the angle of refraction, when the wave $\mathrm{RO}y'$, to which OM is perpendicular, is supposed to have passed into the crystal out of an ordinary medium in which the velocity of propagation is unity. The angles of incidence and refraction are understood to be the angles which the incident and refracted waves respectively make with the refracting surface of the crystal.

The wave-surface and the index-surface have the same geometrical properties, since they are both biaxal surfaces. Let us consider the former, which is generated by the ellipsoid whose semiaxes are a, b, c; and let us conceive this ellipsoid to be intersected by a concentric sphere of which the radius is r. Then the equations of the ellipsoid and the sphere being respectively

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad \frac{x^2+y^2+z^2}{r^2}=1,$$

---

* See Transactions of the Royal Irish Academy, vol. xvii. p. 244.

† Ibid. vol. xvi. p. 68.

§ Ibid, vol. xviii. p. 38. I had previously called it the *surface of refraction*, vol. xvii. p. 252.

we get, by subtracting the one from the other,

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0, \tag{10}$$

for the equation of the cone A which has its vertex at O, and passes through the curve of intersection. If OQ be equal to r, it will be a side of this cone; and a plane touching the cone along OQ will make in the ellipsoid a section of which OQ will be a semiaxis; so that OT will be perpendicular to that plane, and equal in length to r. Therefore, as OQ describes the cone A, the right line OT describes another cone B reciprocal to A, and the point T describes the curve in which the wave-surface is intersected by the sphere above-mentioned; this curve being a *spherical ellipse*, reciprocal to that which the point Q describes on the surface of the ellipsoid. The equation of the cone B is found from that of A, by changing the coefficients of the squares of the variables into their reciprocals, and is therefore

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0, \tag{11}$$

which, of course, is also the equation of the wave-surface, if r be supposed to be the radius drawn from O to the point whose coordinates are x, y, z. Combining this equation with that of the sphere, we have

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1, \tag{12}$$

which represents a hyperboloid passing through the common intersection of the sphere, the cone B, and the wave-surface.

Since the differences between the coefficients of the squares of the variables in the equation (10) are the same as the corresponding differences in the equation of the ellipsoid, the cone A has its planes of circular section coincident with those of the ellipsoid. The cone B, being reciprocal to A, has therefore its focal lines perpendicular to the circular sections of the ellipsoid. These focal lines are consequently the *nodal diameters** of the wave-surface, that is, the diameters which pass through the points where the two sheets of that surface intersect each other.

If the direction of OT cut the other sheet of the wave-surface in T′, and if two radii of constant lengths, equal to OT and OT′ respectively, revolve within the surface, the cones B and B′ described by these radii will intersect each other at right angles, since they have the same focal lines. And supposing the axis of y to be the mean axis of the ellipsoid, so that the nodal diameters lie in the plane of xz, the axis of x will lie within one of the cones, as B, and the axis of z within the other cone B′. Now the angle contained by the two sides of either cone, which lie in the plane of xz, is given by the angles θ and θ' which the direction of the right line OTT′ makes with the nodal diameters; because the angles which any side of a cone makes with its focal lines have a constant sum, or a constant difference, according to the way in which they are reckoned. But if the angles θ and θ' be reckoned (as they may be) so that their sum shall be equal to the angle contained by the two sides of the cone B which are in the plane of xz, their difference will be equal to the angle contained by the two sides of the cone B′ which are in the same plane; the contained angle, in each case, being that which is bisected by the axis of x. Therefore, the lengths OT and OT′, which we denote by r and r', are equal to two radii of the ellipse whose equation is

---

* Ibid. vol. xvii. p. 247.

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

these radii making with the axis of z the angles $\frac{1}{2}(\theta + \theta')$ and $\frac{1}{2}(\theta - \theta')$ respectively. Hence

$$\begin{aligned}\frac{1}{r^2} &= \frac{\sin^2 \frac{1}{2}(\theta+\theta')}{a^2} + \frac{\cos^2 \frac{1}{2}(\theta+\theta')}{c^2} = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) \cos(\theta+\theta'),\\ \frac{1}{r'^2} &= \frac{\sin^2 \frac{1}{2}(\theta-\theta')}{a^2} + \frac{\cos^2 \frac{1}{2}(\theta-\theta')}{c^2} = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) - \frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right) \cos(\theta-\theta').\end{aligned} \tag{13}$$

These formulæ give the two velocities of propagation along a *ray* which makes the angles θ, θ' with the nodal diameters. Subtracting them, we have

$$\frac{1}{r^2} - \frac{1}{r'^2} = \left(\frac{1}{a^2} - \frac{1}{c^2}\right) \sin\theta \sin\theta'. \tag{14}$$

All the preceding equations, relative to the wave-surface, may be transferred to the index-surface, by changing the quantities a, b, c into their reciprocals. For example, if the normal to a wave make the angles θ_0, θ_1 with the nodal diameters of the index-surface, the formulæ (13) give

$$\begin{aligned} s^2 &= \tfrac{1}{2}(a^2 + c^2) - \tfrac{1}{2}(a^2 - c^2)\cos(\theta_0 + \theta_1), \\ s'^2 &= \tfrac{1}{2}(a^2 + c^2) - \tfrac{1}{2}(a^2 - c^2)\cos(\theta_0 - \theta_1)\,; \end{aligned} \tag{15}$$

observing that s and s', the two normal velocities of propagation, are the reciprocals of the radii of this surface which coincide with the wave-normal. Subtracting these expressions, we get

$$s^2 - s'^2 = (a^2 - c^2)\sin\theta_0 \sin\theta_1. \tag{16}$$

As the position of the tangent plane, at any point T of a biaxal surface, depends on the position of the axes of the section QOy' made in the generating ellipsoid by a plane perpendicular to OT, it is obvious that when this section is a circle, that is, when the point T is a node of the surface, the position of the tangent plane is indeterminate, like that of the axes of the section; and it is easy to show that the cone which that plane touches in all its positions is of the second order. Again, when the section ROy' of the reciprocal ellipsoid is a circle, the right line OS is given both in position and length; and the tangent plane, which cuts OS in S, is fixed; but the point of contact T is not fixed, since the semiaxis OR, to which the right line ST is parallel, may be any radius of the circle ROy'. In this case, the point T describes a curve in the tangent plane, and this curve is found to be a circle. But both these cases have been fully discussed elsewhere.*

SECT. V.—CONDITIONS TO BE SATISFIED WHEN LIGHT PASSES OUT OF ONE MEDIUM INTO ANOTHER.—REMARKABLE CIRCUMSTANCE CONNECTED WITH THEM.—RELATIONS AMONG THE TRANSVERSALS OF THE INCIDENT, REFLECTED, AND REFRACTED RAYS.

Now let light pass out of one medium into another, suppose out of an ordinary into a doubly refracting medium; and taking the origin of rectangular coordinates x_0, y_0, z_0 at a point O on the surface which separates the two media, let this surface be the plane of $x_0\,y_0$. Then if the components of the displacement of a particle whose initial coordinates are x_0, y_0, z_0 be denoted by ξ'_0, η'_0, ζ'_0

---

* Ibid. vol. xvii. pp. 245, 260.

when the particle is in the first medium, and by ξ_0'', η_0'', ζ_0'' when it is in the second, the equation (1), adapted to the present case, will be

$$\iiint dx_0 dy_0 dz_0 \left(\frac{d^2\xi_0'}{dt^2}\delta\xi_0' + \frac{d^2\eta_0'}{dt^2}\delta\eta_0' + \frac{d^2\zeta_0'}{dt^2}\delta\zeta_0'\right) + \iiint dx_0 dy_0 dz_0 \left(\frac{d^2\xi_0''}{dt^2}\delta\xi_0'' + \frac{d^2\eta_0''}{dt^2}\delta\eta_0'' + \frac{d^2\zeta_0''}{dt^2}\delta\zeta_0''\right)$$
$$= \iiint dx_0 dy_0 dz_0 \delta \mathrm{V}' + \iiint dx_0 dy_0 dz_0 \delta \mathrm{V}''; \qquad (17)$$

wherein $\delta\mathrm{V}'$ and $\delta\mathrm{V}''$ are the respective values of $\delta\mathrm{V}$ for the two media, which are conceived to extend indefinitely on each side of the plane of $x_0 y_0$; that plane being an upper limit of the integrations relative to one medium, and a lower limit of the integrations relative to the other. Each medium is conceived to be occupied by systems of plane waves, the first by incident and reflected waves, the second by refracted waves; and, except where they are bounded by the plane of $x_0 y_0$, these waves are regarded as unlimited in extent.

For the ordinary medium, if we put

$$\mathrm{X}_0' = \frac{d\eta_0'}{dz_0} - \frac{d\zeta_0'}{dy_0}, \qquad \mathrm{Y}_0' = \frac{d\zeta_0'}{dx_0} - \frac{d\xi_0'}{dz_0}, \qquad \mathrm{Z}_0' = \frac{d\xi_0'}{dy_0} - \frac{d\eta_0'}{dx_0},$$

and suppose the velocity of propagation to be unity, we have*

$$-\delta\mathrm{V}' = \mathrm{X}_0'\left(\frac{d\delta\eta_0'}{dz_0} - \frac{d\delta\zeta_0'}{dy_0}\right) + \mathrm{Y}_0'\left(\frac{d\delta\zeta_0'}{dx_0} - \frac{d\delta\xi_0'}{dz_0}\right) + \mathrm{Z}_0'\left(\frac{d\delta\xi_0'}{dy_0} - \frac{d\delta\eta_0'}{dx_0}\right).$$

For the crystallized medium, if its principal axes be those of x, y, z, the value of $\delta\mathrm{V}''$ will be the same as that of $\delta\mathrm{V}$ in formula (3); but instead of the variations of ξ, η, ζ, we must use those of $\xi_0'', \eta_0'', \zeta_0''$. Denoting the cosines of the angles which the principal axes respectively make with the axis of x_0 by l, m, n; with the axis of y_0 by l', m', n'; with the axis of z_0 by l'', m'', n''; and putting

$$\mathrm{X}_0'' = \frac{d\eta_0''}{dz_0} - \frac{d\zeta_0''}{dy_0}, \qquad \mathrm{Y}_0'' = \frac{d\zeta_0''}{dx_0} - \frac{d\xi_0''}{dz_0}, \qquad \mathrm{Z}_0'' = \frac{d\xi_0''}{dy_0} - \frac{d\eta_0''}{dx_0},$$
$$\delta\mathrm{X}_0'' = \frac{d\delta\eta_0''}{dz_0} - \frac{d\delta\zeta_0''}{dy_0}, \qquad \delta\mathrm{Y}_0'' = \frac{d\delta\zeta_0''}{dx_0} - \frac{d\delta\xi_0''}{dz_0}, \qquad \delta\mathrm{Z}_0'' = \frac{d\delta\xi_0''}{dy_0} - \frac{d\delta\eta_0''}{dx_0};$$

---

* It is assumed here, and in what follows, that when there are two or more coexisting waves in a given medium, the form of the function V is the same as for a single wave, provided the displacements which enter into the function be the resultants of the displacements due to each wave separately. This, however, ought evidently to be the case, in order that the principle of the superposition of vibrations may hold good.

we have

$$\delta \mathrm{X} = l\delta \mathrm{X}_0'' + l'\delta \mathrm{Y}_0'' + l''\delta \mathrm{Z}_0'',$$

$$\delta \mathrm{Y} = m\delta \mathrm{X}_0'' + m'\delta \mathrm{Y}_0'' + m''\delta \mathrm{Z}_0'',$$

$$\delta \mathrm{Z} = n\delta \mathrm{X}_0'' + n'\delta \mathrm{Y}_0'' + n''\delta \mathrm{Z}_0''.$$

These expressions for $\delta \mathrm{X}$, $\delta \mathrm{Y}$, $\delta \mathrm{Z}$ having been written in formula (3), the resulting value of $\delta V''$, as well as the above value of $\delta V'$, is to be substituted in the equation (17), and then the right-hand member of that equation is to be integrated by parts, in order to get rid of the differential coefficients of the variations. When this operation is performed, the triple integrals on one side of the equation will be equal to those on the other; and by equating the coefficients of the corresponding variations in each medium, we should get the laws of propagation in each. But we are not now considering these laws, and we need only attend to the double integrals produced by the operation aforesaid. The double integrals are together equal to zero; but we are concerned only with that part of them which relates to the common limit of the media, the plane of $x_0\, y_0$; and this part must be separately equal to zero, since the conditions to be fulfilled at the plane of $x_0\, y_0$ are independent of any thing that might take place at other limiting surfaces, if such were supposed to exist. Collecting therefore the terms produced by integrating with respect to z_0, and observing that a negative sign must be interposed between those which belong to different media, we get

$$\iint dx_0\, dy_0 (\mathrm{Y}_0' \,\delta\xi_0' - \mathrm{X}_0' \,\delta\eta_0') - \iint dx_0\, dy_0 (\mathrm{Q}\delta\xi_0'' - \mathrm{P}\delta\eta_0'') = 0, \qquad (18)$$

where

$$\mathrm{P} = a^2 l\mathrm{X} + b^2 m\mathrm{Y} + c^2 n\mathrm{Z}, \qquad \mathrm{Q} = a^2 l'\mathrm{X} + b^2 m'\mathrm{Y} + c^2 n'\mathrm{Z}. \qquad (19)$$

In each of these equations it is understood that $z_0 = 0$. But when $z_0 = 0$, we have obviously

$$\xi_0' = \xi_0'', \qquad \eta_0' = \eta_0'', \qquad (20)$$

and therefore

$$\delta\xi_0' = \delta\xi_0'', \qquad \delta\eta_0' = \delta\eta_0'';$$

so that the equation (18) becomes

$$\iint dx_0\, dy_0 \{ (\mathrm{Y}_0' - \mathrm{Q})\, \delta\xi_0' - (\mathrm{X}_0' - \mathrm{P})\, \delta\eta_0' \} = 0,$$

which, as the variations $d\xi_0'$ and $d\eta_0'$ are arbitrary and independent, is equivalent to the two equations

$$X_0' = P, \qquad Y_0' = Q. \tag{21}$$

Thus, to find the relations which subsist among the vibrations incident, reflected, and refracted, at the common surface of two media, we have four conditions, expressed by the equations (20) and (21) ; and these conditions are sufficient to determine the reflected and refracted vibrations, when the incident vibration is given. But though, by the nature of the question, four conditions only are required for its solution, there remains another condition which ought to be satisfied ; for we ought evidently to have

$$\zeta_0' = \zeta_0'', \text{ when } z_0 = 0. \tag{22}$$

This condition is apparently independent of the rest; but it cannot really be so, if the preceding theory is consistent with itself. We shall accordingly see, in what follows, that the last condition is included in the other four ; which is a remarkable circumstance, and a singular confirmation of the theory.*

As the incident and reflected waves coexist in the first medium, and two sets of refracted waves in the second, the resolved displacements, and all the quantities which depend upon them, are composed of two parts, due to the coexisting waves. Let the point O be, for each set of waves, the origin of a system of rectangular coordinates, which we shall call x_1, y_1, z_1 for the incident, and x_1', y_1', z_1' for the reflected wave, the axes of z_1 and z_1' being perpendicular to the respective waves, and their positive directions being those of propagation. Let the displacements in these waves be parallel to y_1, y_1', and be denoted by η_1, η_1', respectively. Then if the axes of x_0, y_0, z_0 make with the axis of x_1 the angles $\alpha_1, \beta_1, \gamma_1$, and with the axis of x_1' the angles $\alpha_1', \beta_1', \gamma_1'$, we have, by the formulæ (F),

$$X_0' = \frac{d\eta_1}{dz_1}\cos\alpha_1 + \frac{d\eta_1'}{dz_1'}\cos\alpha_1', \qquad Y_0' = \frac{d\eta_1}{dz_1}\cos\beta_1 + \frac{d\eta_1'}{dz_1'}\cos\beta_1'. \tag{23}$$

---

* In considering the question of reflexion at the common surface of two ordinary media (Mémoires de l'Institut, tom. xi., p. 396), Fresnel assumes the conditions (20); but his other suppositions violate the condition (22). In fact, this last condition is inconsistent with the supposition that, in a polarized ray, the direction of the vibrations is perpendicular to the plane of polarization. See the Transactions of the Royal Irish Academy, vol. xviii. p. 32.

Again, let the coordinates x_2, y_2, z_2 have reference to one set of refracted waves, and x'_2, y'_2, z'_2 to the other, the axes of z_2 and z'_2 being perpendicular to the respective waves, and their positive directions being those in which the waves are propagated. Suppose the displacements to be parallel to y_2, y'_2, and to be denoted by η_2, η'_2 respectively. Then if the axes of x, y, z make with the axis of x_2 the angles $\alpha_{(2)}$, $\beta_{(2)}$, $\gamma_{(2)}$, and with the axis of x'_2 the angles $\alpha'_{(2)}$, $\beta'_{(2)}$, $\gamma'_{(2)}$, we have, by the formulæ (F),

$$\mathrm{X} = \frac{d\eta_2}{dz_2}\cos\alpha_{(2)} + \frac{d\eta'_2}{dz'_2}\cos\alpha'_{(2)},$$

$$\mathrm{Y} = \frac{d\eta_2}{dz_2}\cos\beta_{(2)} + \frac{d\eta'_2}{dz'_2}\cos\beta'_{(2)},$$

$$\mathrm{Z} = \frac{d\eta_2}{dz_2}\cos\gamma_{(2)} + \frac{d\eta'_2}{dz'_2}\cos\gamma'_{(2)};$$

and thence, by the relations (19),

$$\begin{aligned}\mathrm{P} &= \frac{d\eta_2}{dz_2}(a^2l\cos\alpha_{(2)} + b^2m\cos\beta_{(2)} + c^2n\cos\gamma_{(2)})\\ &\quad + \frac{d\eta'_2}{dz'_2}(a^2l\cos\alpha'_{(2)} + b^2m\cos\beta'_{(2)} + c^2n\cos\gamma'_{(2)}),\\ \mathrm{Q} &= \frac{d\eta_2}{dz_2}(a^2l'\cos\alpha_{(2)} + b^2m'\cos\beta_{(2)} + c^2n'\cos\gamma_{(2)})\\ &\quad + \frac{d\eta'_2}{dz'_2}(a^2l'\cos\alpha'_{(2)} + b^2m'\cos\beta'_{(2)} + c^2n'\cos\gamma'_{(2)}).\end{aligned}$$

Suppose the ellipsoid which generates the wave-surface of the second medium to have its centre at O, and to be touched in the points Q and Q′ by two planes which cut the axes of x_2 and x'_2 perpendicularly in the points P and P′; the lengths OP and OP′ being expressed by s, s', and the lengths OQ and OQ′ by r, r'. Let the axes of x_0, y_0, z_0 make with the direction of OQ the angles α_2, β_2, γ_2, and with the direction of OQ′ the angles α'_2, β'_2, γ'_2. Then, from the equation (H) in Lemma IV., it is manifest that

$$\begin{aligned}a^2l\cos\alpha_{(2)} + b^2m\cos\beta_{(2)} + c^2n\cos\gamma_{(2)} &= rs\cos\alpha_2,\\ a^2l\cos\alpha'_{(2)} + b^2m\cos\beta'_{(2)} + c^2n\cos\gamma'_{(2)} &= r's'\cos\alpha'_2,\\ a^2l'\cos\alpha_{(2)} + b^2m'\cos\beta_{(2)} + c^2n'\cos\gamma_{(2)} &= rs\cos\beta_2,\\ a^2l'\cos\alpha'_{(2)} + b^2m'\cos\beta'_{(2)} + c^2n'\cos\gamma'_{(2)} &= r's'\cos\beta'_2.\end{aligned}$$

Hence,

$$\mathrm{P} = \frac{d\eta_2}{dz_2} rs \cos a_2 + \frac{d\eta_2'}{dz_2'} r's' \cos a_2', \qquad \mathrm{Q} = \frac{d\eta_2}{dz_2} rs \cos \beta_2 + \frac{d\eta_2'}{dz_2'} r's' \cos \beta_2'. \quad (24)$$

The quantities s, s' are, as appears by the last section, the normal velocities with which the two sets of refracted waves are propagated. The velocity with which the incident and reflected waves are propagated is taken as unity. Therefore, if τ_1, τ_1' be the transversals, or amplitudes of vibration in the incident and reflected waves, and τ_2, τ_2' the transversals of the refracted waves, the lengths of the latter waves being denoted by λ_2, λ_2', and the length of an incident or reflected wave by λ_1, and if we put

$$\begin{aligned} \phi_1 &= \frac{2\pi}{\lambda_1}(t - z_1 + \nu_1), & \phi_1' &= \frac{2\pi}{\lambda_1}(t - z_1' + \nu_1'), \\ \phi_2 &= \frac{2\pi}{\lambda_2}(st - z_2 + \nu_2), & \phi_2' &= \frac{2\pi}{\lambda_2'}(s't - z_2' + \nu_2'), \end{aligned} \quad (25)$$

where ν_1, ν_1', ν_2, ν_2' are constants, and π is the ratio of the circumference to the diameter of a circle, we may write

$$\eta_1 = \tau_1 \cos \phi_1, \quad \eta_1' = \tau_1' \cos \phi_1', \quad \eta_2 = \tau_2 \cos \phi_2, \quad \eta_2' = \tau_2' \cos \phi_2'. \quad (26)$$

By means of these values the formulæ (23) and (24) become

$$\begin{aligned} \mathrm{X}_0' &= \frac{2\pi}{\lambda_1}(\tau_1 \cos a_1 \sin \phi_1 + \tau_1' \cos a_1' \sin \phi_1'), \\ \mathrm{Y}_0' &= \frac{2\pi}{\lambda_1}(\tau_1 \cos \beta_1 \sin \phi_1 + \tau_1' \cos \beta_1' \sin \phi_1'), \\ \mathrm{P} &= 2\pi\left(\frac{rs}{\lambda_2}\tau_2 \cos a_2 \sin \phi_2 + \frac{r's'}{\lambda_2'}\tau_2' \cos a_2' \sin \phi_2'\right), \\ \mathrm{Q} &= 2\pi\left(\frac{rs}{\lambda_2}\tau_2 \cos \beta_2 \sin \phi_2 + \frac{r's'}{\lambda_2'}\tau_2' \cos \beta_2' \sin \phi_2'\right). \end{aligned} \quad (27)$$

The angles ϕ_1, ϕ_1', ϕ_2, ϕ_2' are the phases of vibration in the different waves at the time t. To see how they depend on the coordinates x_0, y_0, z_0, conceive the axis of z_0 to be directed from O towards the interior of the second medium, and the axis of x_0 to lie in the plane of incidence, so that the positive directions of z_1, z_2, z_2' may lie within the angle made by the positive directions of x_0 and z_0, while the positive direction of z'_1 lies within the angle made by the positive direction of x_0 and the negative direction of z_0. Let i_1 be the angle of incidence, and i_2, i'_2 the angles of refraction; then

$$\begin{aligned} z_1 &= x_0 \sin i_1 + z_0 \cos i_1, \qquad & z'_1 &= x_0 \sin i_1 - z_0 \cos i_1, \\ z_2 &= x_0 \sin i_2 + z_0 \cos i_2, \qquad & z'_2 &= x_0 \sin i'_2 + z_0 \cos i'_2. \end{aligned} \tag{28}$$

These values are to be written in the expressions (25). They show that the phases, and therefore the displacements, are independent of y_0.

Since the conditions relative to the plane of $x_0\, y_0$ must hold at every instant of time, and for every point of that plane, the coefficients of t, as well as those of x_0, in the values of the different phases, must be identical; so that we must have

$$\frac{1}{\lambda_1} = \frac{s}{\lambda_2} = \frac{s'}{\lambda'_2}, \qquad \frac{\sin i_1}{\lambda_1} = \frac{\sin i_2}{\lambda_2} = \frac{\sin i'_2}{\lambda'_2}. \tag{29}$$

Therefore, when $z_0 = 0$, the supposition

$$\nu_1 = \nu'_1 = \nu_2 = \nu'_2 \tag{30}$$

renders the phases identical, independently of t and x_0. And, from the form of the equations of condition, it is easy to see that this supposition is necessary; because the equations (20), when the values (26) are substituted in them, contain only the cosines of the phases; and the equations (21), when the values (27) are substituted in them, contain only the sines of the phases. Making the latter substitution, and attending to the relations just mentioned, we find

$$\begin{aligned} \tau_1 \cos \alpha_1 + \tau'_1 \cos \alpha'_1 &= r\tau_2 \cos \alpha_2 + r'\tau'_2 \cos \alpha'_2, \\ \tau_1 \cos \beta_1 + \tau'_1 \cos \beta'_1 &= r\tau_2 \cos \beta_2 + r'\tau'_2 \cos \beta'_2. \end{aligned} \tag{31}$$

In these equations, the angles by whose cosines each transversal is multiplied are the angles which a plane, passing through the directions of that transversal and of the corresponding *ray*, makes with the planes of $y_0\, z_0$ and $x_0\, z_0$. This is evident with regard to the incident and reflected rays. And if we refer to the diagram in the preceding section, it will also be evident with regard to the refracted rays; for OQ is perpendicular to the transversal τ_2, and to the right line OT, which is the direction of the corresponding ray.

Taking O for the point of incidence, let right lines proceeding from it represent the different rays; and let the length of each ray, measured from O in the direction of propagation, be assumed proportional to the velocity with which the light it propagated along it. Through the extremity of each ray conceive its transversal to be drawn, and let the transversals so drawn have their *moments* taken, with respect to the point O, as if they represented forces applied to a rigid body. The length of the incident or reflected ray being considered as unity, the lengths of the refracted rays (as appears by the last Section) are r and r' respectively. Hence, as each transversal is perpendicular to its ray, the moments of the incident and reflected transversals are proportional to τ_1, τ_1', and the moments of the refracted transversals to $r\tau_2$, $r'\tau_2'$ respectively. The equations (31) therefore signify that when the moments are projected, either upon the plane of $y_0 z_0$, or upon the plane of $x_0 z_0$, the total projected moments are the same for the two media; or that, if the transversals themselves be projected on either of these planes, the moments of the projections of the incident and reflected transversals are together equal to the moments of the projections of the refracted transversals.

But the second of the equations (31) has another signification. For if the transversals applied at the extremities of the refracted rays be projected on the plane of $x_0 z_0$, which is the plane of incidence, and contains the axes of z_2 and z_2', the projections will be perpendicular to these axes, since the transversals themselves are perpendicular to them; and the distances of the projections from the point O will be proportional to s and s', or, by the relations (29), to $\sin i_2$ and $\sin i_2'$; so that if θ_2 and θ_2' be the angles which the transversals make with the plane of incidence, the moments of the projections will be represented by $\tau_2 \cos\theta_2 \sin i_2$ and $\tau_2' \cos\theta_2' \sin i_2'$. At the same time, if θ_1 and θ_1' be the angles which the incident and reflected transversals make with the plane of incidence, the moments of the corresponding projections of these transversals will evidently be represented by $\tau_1 \cos\theta_1 \sin i_1$ and $-\tau_1' \cos\theta_1' \sin i_1$; the latter quantity being taken with a negative sign, because the extremity of the reflected ray, where the transversal τ_1' is applied, lies in the first medium, while the extremities of the incident and refracted rays lie in the second, and it is supposed that when any of the angles θ_1, θ_1', θ_2, θ_2' is zero, the direction of the corresponding transversal makes an acute angle with the axis of x_0. Hence we have

$$\tau_1 \cos\theta_1 \sin i_1 - \tau_1' \cos\theta_1' \sin i_1 = \tau_2 \cos\theta_2 \sin i_2 + \tau_2' \cos\theta_2' \sin i_2';$$

an equation which expresses that if each transversal be projected upon the axis of z_0, the sum of the projections of the incident and reflected transversals will be equal to the sum of the projections of the refracted transversals. Therefore, since the phases of the different vibrations are identical when $z_0 = 0$, the condition (22) is fulfilled, as it ought to be.

On account of this identity of phases, it follows from the conditions (20) and (22), that if the transversals be drawn through the point O, and those which belong to each medium be compounded like forces acting at a point, their resultants will be the same; that is, the resultant of the incident and reflected transversals will be the same as the resultant of the refracted transversals.

Hence, recollecting what has been proved respecting the moments of the transversals applied at the extremities of the rays, we have the following theorem :

Supposing the length of each ray, measured from the point of incidence and in the direction of propagation, to be taken proportional to the velocity with which the light is propagated along it, and its transversal to be drawn through the extremity of this length, the incident and reflected transversals having their proper directions, but the refracted transversals having their directions reversed ; if all the transversals so drawn be compounded like forces applied to a rigid body, their resultant will be a couple, lying in a plane parallel to the plane which separates the two media.

This theorem affords a complete solution of the question of reflexion and refraction.* Expressed analytically it gives five equations, of which four are independent.

To apply the preceding results to a simple case, suppose the second medium, as well as the first, to be an ordinary one. We have then only one refracted ray, and one refracted transversal τ_2.

1°. When the incident ray is polarized in the plane of incidence, the transversals are all in that plane; and as they are perpendicular to the rays, and the refracted transversal is the resultant of the other two, we have evidently

---

* The same theorem applies to the other case of reflexion and refraction, when a ray which has entered the crystal emerges from it into an ordinary medium, undergoing double reflexion at the surface where it emerges. In fact, the conditions (20) and (21) hold good whether the ordinary medium is the first or the second ; and in the latter case, as well as in the former, it may be shown that the condition (22) is fulfilled, and that the theorem above-mentioned is true.

$$\tau_1' = \tau_1 \frac{\sin(i_1 - i_2)}{\sin(i_1 + i_2)}, \qquad \tau_2 = \tau_1 \frac{\sin 2i_1}{\sin(i_1 + i_2)}. \tag{32}$$

2°. When the incident ray is polarized perpendicularly to the plane of incidence, the transversals are all perpendicular to that plane. Taking $2\sin i_1$ to represent the length of the incident or reflected ray, the proportional length of the refracted ray is $2\sin i_2$, and the projections of these lengths on the plane of $y_0 z_0$ are $2\sin i_1 \cos i_1$ and $2\sin i_2 \cos i_2$, or $\sin 2i_1$ and $\sin 2i_2$. The transversals applied at the extremities of the rays are not altered by being projected on the plane of $y_0 z_0$; therefore the moments of the incident, reflected, and refracted transversals, projected on this plane, are represented by the quantities $\tau_1 \sin 2i_1$, $-\tau_1' \sin 2i_1$, and $\tau_2 \sin 2i_2$ respectively. Equating the last moment to the sum of the other two, and the refracted transversal to the sum of the other two transversals, we get

$$(\tau_1 - \tau_1') \sin 2i_1 = \tau_2 \sin 2i_2, \qquad \tau_1 + \tau_1' = \tau_2;$$

and thence

$$\tau_1' = \tau_1 \frac{\tan(i_1 - i_2)}{\tan(i_1 + i_2)}, \qquad \tau_2 = \tau_1 \frac{\sin 2i_1}{\sin(i_1 + i_2)\cos(i_1 - i_2)}. \tag{33}$$

This case has been considered by Fresnel. The relative magnitudes of the incident and reflected transversals, as given by him, are in accordance* with the formulæ (32) and (33); but with respect to the refracted transversals, his results do not agree with the formulæ.

SECT. VI.—PRESERVATION OF *VIS VIVA*—THEOREM OF THE POLAR PLANE—CONCLUSION.

Returning to the general question, if we resolve the transversals parallel to the axes of x_0, y_0, z_0, and equate the sums of the parallel components in one medium to the corresponding sums in the other, we get the three conditions

---

* There is, however, a difference as to the relative directions of the incident and reflected transversals. When the second medium is the denser, and the incidence is perpendicular, these transversals, according to the present theory, have the same direction, but according to Fresnel they have opposite directions.

$$\begin{gathered}(\tau_1 \cos\theta_1 + \tau_1' \cos\theta_1') \cos i_1 = \tau_2 \cos\theta_2 \cos i_2 + \tau_2' \cos\theta_2' \cos i_2', \\ \tau_1 \sin\theta_1 + \tau_1' \sin\theta_1' = \tau_2 \sin\theta_2 + \tau_2' \sin\theta_2', \\ (\tau_1 \cos\theta_1 - \tau_1' \cos\theta_1') \sin i_1 = \tau_2 \cos\theta_2 \sin i_2 + \tau_2' \cos\theta_2' \sin i_2'.\end{gathered} \tag{34}$$

A fourth condition is supplied by the first of the equations (31), in which equation we have to write

$$\cos\alpha_1 = \sin\theta_1 \cos i_1, \qquad \cos\alpha_1' = -\sin\theta_1' \cos i_1,$$

and to substitute similar expressions for $\cos\alpha_2$, $\cos\alpha_2'$.

The right line OQ is perpendicular to the transversal τ_2 and to the ray OT. The cosines of the angles α_2, β_2, γ_2 may therefore be found by means of the cosines of the angles which the transversal and the ray make with the axes of x_0, y_0, z_0.

The cosines of the angles which the transversal τ_2 makes with these axes are respectively

$$\cos\theta_2 \cos i_2, \qquad \sin\theta_2, \qquad -\cos\theta_2 \sin i_2.$$

As the plane which passes through the ray and the wave-normal OS is perpendicular to the transversal τ_2, this plane makes with the plane of incidence an angle equal to $90° + \theta_2$ or $90° - \theta_2$. Let a sphere, having its centre at O, be intersected in the points S_0, T_0 by the right lines OS, OT, and in the points X_0, Y_0, Z_0 by the axes of x_0, y_0, z_0; and conceive the points T_0 and Y_0 to be at the same side of the plane of $x_0\, z_0$, the spherical angle $T_0S_0X_0$ being $90° + \theta_2$, and the spherical angle $T_0S_0Z_0$ being $90° - \theta_2$. Let ϵ be the angle which the ray makes with the wave-normal. Then, the angles which the ray makes with the axes of coordinates being measured by the arcs T_0X_0, T_0Y_0, T_0Z_0, the cosines of these angles respectively are

$$\sin i_2 \cos\epsilon - \sin\theta_2 \cos i_2 \sin\epsilon, \qquad \cos\theta_2 \sin\epsilon, \qquad \cos i_2 \cos\epsilon + \sin\theta_2 \sin i_2 \sin\epsilon.$$

Hence, as the transversal is at right angles to the ray, we have, by Lemma I,

$$\begin{gathered}\cos\alpha_2 = \sin i_2 \sin\epsilon + \sin\theta_2 \cos i_2 \cos\epsilon, \qquad \cos\beta_2 = -\cos\theta_2 \cos\epsilon, \\ \cos\gamma_2 = \cos i_2 \sin\epsilon - \sin\theta_2 \sin i_2 \cos\epsilon,\end{gathered} \tag{35}$$

In like manner, putting ϵ' for the angle which the other refracted ray makes with its wave-normal, we have

$$\begin{gathered}\cos\alpha_2' = \sin i_2' \sin\epsilon' + \sin\theta_2' \cos i_2' \cos\epsilon', \qquad \cos\beta_2' = -\cos\theta_2' \cos\epsilon', \\ \cos\gamma_2' = \cos i_2' \sin\epsilon' - \sin\theta_2' \sin i_2' \cos\epsilon'.\end{gathered} \tag{36}$$

If we substitute, in the first of the equations (31), the values just given for $\cos a_2$, $\cos a'_2$, along with the above values of $\cos a_1$, $\cos a'_1$, and attend to the relations

$$r\cos\epsilon = s, \qquad r'\cos\epsilon' = s', \qquad \sin i_2 = s\sin i_1, \qquad \sin i'_2 = s'\sin i_1, \tag{37}$$

we find, after multiplying all the terms of the equation by $\sin i_1$,

$$\begin{gathered}(\tau_1\sin\theta_1 - \tau'_1\sin\theta'_1)\sin i_1\cos i_1 = \\ \tau_2(\sin\theta_2\sin i_2\cos i_2 + \sin^2 i_2\tan\epsilon) + \tau'_2(\sin\theta'_2\sin i'_2\cos i'_2 + \sin^2 i'_2\tan\epsilon').\end{gathered} \tag{38}$$

Joining this equation to the equations (34) we have all the conditions that are necessary for the solution of the question.

Multiplying the first of the equations (34) by the third, also the second of these equations by the equation (38), adding the products together, and then dividing by $\sin i_1$, we obtain

$$\mu_1(\tau_1^2 - \tau_1'^2) = \mu_2\tau_2^2 + \mu'_2\tau_2'^2 + \text{M}\tau_2\tau'_2, \tag{39}$$

where we have put

$$\begin{gathered}\mu_1 = \cos i_1, \qquad \mu_2 = s(\cos i_2 + \sin\theta_2\sin i_2\tan\epsilon), \\ \mu'_2 = s'(\cos i'_2 + \sin\theta'_2\sin i'_2\tan\epsilon'), \\ \text{M}\sin i_1 = \sin(i_2 + i'_2)\{\cos\theta_2\cos\theta'_2 + \sin\theta_2\sin\theta'_2\cos(i_2 - i'_2)\} \\ + \sin\theta'_2\sin^2 i_2\tan\epsilon + \sin\theta_2\sin^2 i'_2\tan\epsilon'.\end{gathered}$$

The last expression may be put under the form

$$\begin{gathered}\text{M}\sin i_1\sin(i_2 - i'_2) = \\ \sin^2 i_2\{\cos\theta_2\cos\theta'_2 + \sin\theta_2\sin\theta'_2\cos(i_2 - i'_2) + \sin\theta'_2\sin(i_2 - i'_2)\tan\epsilon\} \\ - \sin^2 i'_2\{\cos\theta_2\cos\theta'_2 + \sin\theta_2\sin\theta'_2\cos(i_2 - i'_2) - \sin\theta_2\sin(i_2 - i'_2)\tan\epsilon'\}.\end{gathered} \tag{40}$$

Let the axes of x_0, y_0, z_0 make with the direction of OP the angles $\alpha_{\prime\prime}$, $\beta_{\prime\prime}$, $\gamma_{\prime\prime}$, and with the direction of OP′ the angles $\alpha'_{\prime\prime}$, $\beta'_{\prime\prime}$, $\gamma'_{\prime\prime}$. The cosines of these angles may be found from the expressions (35) and (36) by supposing ϵ and ϵ' to vanish. Therefore

$$\begin{gathered}\cos\alpha_{\prime\prime} = \sin\theta_2\cos i_2, \quad \cos\beta_{\prime\prime} = -\cos\theta_2, \quad \cos\gamma_{\prime\prime} = -\sin\theta_2\sin i_2, \\ \cos\alpha'_{\prime\prime} = \sin\theta'_2\cos i'_2, \quad \cos\beta'_{\prime\prime} = -\cos\theta'_2, \quad \cos\gamma'_{\prime\prime} = -\sin\theta'_2\sin i'_2.\end{gathered} \tag{41}$$

If ω be the angle which OQ makes with OP′, and ω' the angle which OQ′ makes with OP, so that

$$\cos\omega = \cos\alpha_2 \cos\alpha'_{,,} + \cos\beta_2 \cos\beta'_{,,} + \cos\gamma_2 \cos\gamma'_{,,},$$
$$\cos\omega' = \cos\alpha'_2 \cos\alpha_{,,} + \cos\beta'_2 \cos\beta_{,,} + \cos\gamma'_2 \cos\gamma_{,,},$$

we find, by the formulæ (35), (36), (41),

$$\cos\omega = \cos\epsilon\,\{\cos\theta_2 \cos\theta'_2 + \sin\theta_2 \sin\theta'_2 \cos(i_2 - i'_2) + \sin\theta'_2 \sin(i_2 - i'_2)\tan\epsilon\},$$
$$\cos\omega' = \cos\epsilon'\{\cos\theta_2 \cos\theta'_2 + \sin\theta_2 \sin\theta'_2 \cos(i_2 - i'_2) - \sin\theta_2 \sin(i_2 - i'_2)\tan\epsilon'\}.$$

Hence, observing the relations (37), we see that the right-hand member of the equation (40) is equal to the quantity

$$\sin^2 i_1\,(rs\cos\omega - r's'\cos\omega').$$

But, by the property of the ellipsoid (Lemma IV), this quantity is zero ;* therefore $M = 0$, and the equation (39) becomes

$$\mu_1 \tau_1{}^2 = \mu_1 \tau_1'^2 + \mu_2 \tau_2{}^2 + \mu'_2 \tau_2'^2. \qquad (42)$$

On each ray let a length, representing the velocity with which the light is propagated along it, be measured, as before, from the point O. The distances of the plane of $x_0\,y_0$ from the extremities of these lengths will be proportional to the coefficients of the squares of the transversals in the preceding equation. For if we take, on the incident or reflected ray, a length equal to unity, its projection on the axis of z_0 will be cos i_1 or μ_1; and if, on the refracted ray OT, we take a length equal to r, its projection on the same axis will be

$$r\,(\cos i_2 \cos\epsilon + \sin\theta_2 \sin i_2 \sin\epsilon),$$

which is equal to μ_2. Similarly, the length r', assumed on the other refracted ray, will have its projection equal to μ'_2. The quantities by which the squares of the transversals are multiplied, in the equation (42), are therefore the corresponding ethereal volumes† which we may conceive to be put in motion by the different waves ; and as we suppose the density of the ether to be the same in both media, the equation expresses a principle analogous to that of the preservation of *vis viva.**

---

* The equation $M = 0$ is the same as the equation (vii.) in my former paper.—Transactions R. I. A., vol. xviii. p. 52.

† Ibid. p. 48.

By giving a certain direction to the incident transversal, that is, by polarising the incident ray in a certain plane, we may make one of the refracted rays disappear. If OT be the ray which remains, we have $\tau_2' = 0$, and the equations (34) and (38) become

$$\left.\begin{aligned}(\tau_1 \cos \theta_1 + \tau_1' \cos \theta_1') \cos i_1 &= \tau_2 \cos \theta_2 \cos i_2,\\ \tau_1 \sin \theta_1 + \tau_1' \sin \theta_1' &= \tau_2 \sin \theta_2,\\ (\tau_1 \cos \theta_1 - \tau_1' \cos \theta_1') \sin i_1 &= \tau_2 \cos \theta_2 \sin i_2,\\ (\tau_1 \sin \theta_1 - \tau_1' \sin \theta_1') \sin i_1 \cos i_1 &= \tau_2 (\sin \theta_2 \sin i_2 \cos i_2 + \sin^2 i_2 \tan \epsilon).\end{aligned}\right\} \quad (43)$$

In this case, the three transversals are in the same plane, the refracted transversal being the resultant of the other two. Therefore if we find this plane, every thing will be determined.

The axes of x_0, y_0, z_0 make with the incident transversal angles whose cosines are

$$\cos \theta_1 \cos i_1, \qquad \sin \theta_1, \qquad -\cos \theta_1 \sin i_1,$$

and with the reflected transversal angles whose cosines are

$$\cos \theta_1' \cos i_1, \qquad \sin \theta_1', \qquad \cos \theta_1' \sin i_1 ;$$

therefore, by Lemma I, the cosines of the angles which these axes make with a right line perpendicular to the plane of the transversals are proportional to the quantities

$$\sin (\theta_1 + \theta_1') \sin i_1, \qquad -\cos \theta_1 \cos \theta_1' \sin 2i_1, \qquad \sin (\theta_1' - \theta_1) \cos i_1.$$

Now from the product of the first and second of the equations (43), combined with the product of the third and fourth, we find, by the help of the relations (37),

$$2\tau_1 \tau_1' \sin (\theta_1 + \theta_1') \sin i_1 = \tau_2^2 \tan i_1 \cos \theta_2 \{\sin \theta_2 \cos i_2 - s^2 (\sin \theta_2 \cos i_2 + \sin i_2 \tan \epsilon)\}.$$

From the squares of the first and third of those equations we find

$$-2\tau_1 \tau_1' \cos \theta_1 \cos \theta_1' \sin 2i_1 = \tau_2^2 \tan i_1 \cos^2 \theta_2 (s^2 - 1),$$

---

* A similar equation of *vis viva* holds when the light passes out of a crystal into an ordinary medium.

and from the product of the first and fourth, combined with the product of the second and third,

$$2\tau_1\tau_1'\sin(\theta_1'-\theta_1)\cos i_1=\tau_2^2\tan i_1\cos\theta_2\{-\sin\theta_2\sin i_2+s^2(\sin\theta_2\sin i_2-\cos i_2\tan\epsilon)\}.$$

In the three equations just found, the left-hand members are proportional, as we have seen, to the cosines of the angles which a right line perpendicular to the plane of the transversals makes with the axes of coordinates; and the right-hand members, as appears by the formulæ (35) and (41), are proportional to the quantities

$$\frac{\cos\alpha_{\prime\prime}}{s}-r\cos\alpha_2, \qquad \frac{\cos\beta_{\prime\prime}}{s}-r\cos\beta_2, \qquad \frac{\cos\gamma_{\prime\prime}}{s}-r\cos\gamma_2,$$

which are obviously the differences between the corresponding coordinates of the points R and Q. The plane of the transversals is therefore perpendicular to the right line QR, which joins those points.

A plane parallel to the right line TM, and passing through the transversal of the ray OT, is that which I have called the *polar plane* of the ray;* and this plane is perpendicular to QR. Therefore, when there is only one refracted ray, the incident and reflected transversals lie in the polar plane of that ray; and their directions being thus determined, the relative magnitudes of the three transversals are known. In this case the incident and reflected transversals are called *uniradial;* and as each refracted ray in turn may be made to disappear, there are two uniradial directions in the plane of the incident wave, and two in that of the reflected wave.

When the incident transversal is not uniradial, it may be considered as the resultant of two uniradial transversals, each of which will supply a refracted ray, and will produce a uniradial component of the reflected transversal.

It is needless to extend these deductions further. They have been carried far enough to show that the results of the foregoing theory are in perfect accordance with the laws established in my former paper on the subject of crystalline reflexion. The theory itself suggests much matter for consideration; but at present we shall confine ourselves to one remark, which may be necessary to prevent any misconception as to the nature of the foundation on which it stands. In this theory, every thing depends on the form of the function V; and we have seen that, when that form is properly assigned, the laws by which crystals act upon light are included in the general equation of dynamics. This fact is fully proved by the preceding investigations. But the reasoning which has been used to account for the form of the function* is indirect, and cannot be regarded as sufficient, in a mechanical point of view. It is, however, the only kind of reasoning which we are able to employ, as the constitution of the luminiferous medium is entirely unknown.

---

* Ibid, p. 39.

---

* Since this paper was read to the Academy, I have found that the form of the function V is more general than it would seem to be from the mode in which it is here deduced; and I have obtained from it a theory of the Total Reflexion of Light. For a sketch of this theory, see the Proceedings of the Royal Irish Academy, vol. ii. p. 96.